APPLICATION OF DOUBLE KAMAL-SHEHU DECOMPOSITION METHOD TO SOLVE NONLINEAR SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

In this work, we develop a new method to obtain approximate solutions of nonlinear coupled partial differential equations with the help of Double Kamal-Shehu de- composition method. The nonlinear term can easily be handled with the help of Adomian polynomials. The results of the present technique have closed agreement with approximate solutions obtained with the help of Adomian decomposition method (ADM).

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1. Introduction

In the literature, several different transforms are introduced and applied to find the solution of partial differential equations such as Laplace transform [7], Shehu transform [9], Kamal transform [1, 6], and so on. Due to the rapid development in the physical science and engineering models, there are many other integral transforms in the literature. Through these transformations, many problems of engineering and science have been solved. However, it has been found that these transformations remain limited in solving equations containing a nonlinear part. To take advantage of these transformations and use them in solving nonlinear (system) differential equations, researchers in the field of mathematics were guided to the idea of forming them in some ways such as the Adomian decomposition method, see for example [2, 4, 8, 10]. The decomposition method has been shown to solve efficiently, easily and accurately a large class of linear and nonlinear ordinary, partial, deterministic or stochastic differential equations [5, 13]. The Adomian decomposition method is relatively easy to implement, and it can be used with other methods.
The aim of this study is to combine two powerful methods, the Adomian decomposition method and the double Kamal-Shehu transform method to obtain a better method for solving nonlinear system partial differential equations. The modified method is called double Kamal-Shehu transformation decomposition method. We apply our modified method to solve some examples of nonlinear system partial differential equations.

2. Preliminaries

The Kamal transform and Shehu transform are new integral transforms similar to the Laplace transform and other integral transforms that are defined in the time domain.

Definition 2.1. ([6]) Let $\mathcal{A}$ be a function set defined by

$$\mathcal{A} = \left\{ u(x) : \exists M, \gamma_1, \gamma_2 > 0, |u(x)| < Me^{\frac{|x|}{\gamma_j}},\ x \in (-1)^j \times [0, \infty),\ j = 1, 2 \right\},$$

where M is a constant and γ_1, γ_2 are finite constants or infinite.
For a function of exponential order, the single Kamal integral transform of the real continuous function $u(x)$ is defined as follows;

$$K[u(x)] = \int_0^\infty e^{-\frac{x}{\gamma}} u(x) dx = F(\gamma),$$

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where $e^{-\frac{x}{\gamma}}$ is the kernel function and $K[.]$ is the Kamal transform operator.
The inverse Kamal transform is defined by

$$u(x) = K^{-1}[F(\gamma)] = \frac{1}{2\pi i}\int_{\kappa - i\infty}^{\kappa + i\infty} e^{\frac{x}{\gamma}} F(\frac{1}{\gamma}) d\gamma, \ \kappa \geq 0.$$

Definition 2.2. ([3, 9]) Over the set of functions,

$$\mathcal{M} = \Big\{ u(t) : \exists N, \kappa_1, \kappa_2 > 0, |u(t)| < N e^{\frac{|t|}{\kappa_i}}, \ for \ t \in (-1)^i \times [0, \infty), \ i = 1, 2 \Big\},$$

the Shehu transform is defined by

$$S[u(t)] = \int_0^{\infty} e^{-\frac{\delta}{\mu} t} u(t) dt = F(\delta, \mu), \ \delta, \mu > 0,$$

where $e^{-\frac{\delta}{\mu}}$ is the kernel function and S is the operator of Shehu transform.
The inverse Shehu transform is defined by

$$u(t) = S^{-1}[F(\delta, \mu)] = \frac{1}{2\pi i}\int_{\omega - i\infty}^{\omega + i\infty} \frac{1}{\mu} e^{\frac{\delta}{\mu} t} F(\delta, \mu) d\delta, \ \omega \geq 0.$$

Definition 2.3. The double Kamal-Shehu transform of the function $u(x, t)$ of two variables $x > 0$ and $t > 0$ is denoted by $K_x S_t[u(x, t)] = F(\gamma, \delta, \mu)$ and defined as

$$\begin{aligned} K_x S_t[u(x,t)] = \quad & F(\gamma, \delta, \mu) = \int_0^{\infty}\int_0^{\infty} e^{-(\frac{x}{\gamma} + \frac{\delta}{\mu} t)} u(x,t) dx dt \\ = \quad & \lim_{\alpha \to \infty, \beta \to \infty} \int_0^{\alpha}\int_0^{\beta} e^{-(\frac{x}{\gamma} + \frac{\delta}{\mu} t)} u(x,t) dx dt. \end{aligned}$$

It converges if the limit of the integral exists, and diverges if not.

Definition 2.4. The inverse double Kamal-Shehu transform of a function $F(\gamma, \delta, \mu)$ is given by

$$K_x^{-1} S_t^{-1}[F(\gamma, \delta, \mu)] = u(x, t).$$

Equivalently,

$$u(x,t) = K_x^{-1} S_t^{-1}[F(\gamma, \delta, \mu)] = \frac{1}{(2\pi i)^2}\int_{\kappa - i\infty}^{\kappa + i\infty} e^{\frac{x}{\gamma}} d\gamma \int_{\omega - i\infty}^{\omega + i\infty} \frac{1}{\mu} e^{\frac{\delta}{\mu} t} F(\frac{1}{\gamma}, \delta, \mu) d\delta,$$

where κ and ω are real constants.

We recall that, double Kamal-Shehu transform for second partial derivatives property

$$K_xS_t\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right] = \frac{1}{\gamma^2}F(\gamma,\delta,\mu) - \frac{1}{\gamma}S[u(0,t)] - S[u_x(0,t)],$$
$$K_xS_t\left[\frac{\partial^2 u(x,t)}{\partial t^2}\right] = \frac{\delta^2}{\mu^2}F(\gamma,\delta,\mu) - \frac{\delta}{\mu}K[u(x,0)] - K[u_t(x,0)],$$
$$K_xS_t\left[\frac{\partial^2 u(x,t)}{\partial x\partial t}\right] = \frac{\delta}{\gamma\mu}F(\gamma,\delta,\mu) - \frac{1}{\gamma}K[u(x,0)] - S[u_t(0,t)],$$

where $K[.]$ and $S[.]$ denote single Kamal transform and single Shehu transform respectively.

1. Analysis of Double Kamal-Shehu transform Decomposition Method

To give an overview of the method, consider the system of partial differential equations in an operator form as given below:

$$\begin{aligned} L_1u(x,y,t) + R_1(u,v,w) + N_1(u,v,w) = f_1(x,y,t),\\ L_2v(x,y,t) + R_2(u,v,w) + N_2(u,v,w) = f_2(x,y,t),\\ L_3w(x,y,t) + R_3(u,v,w) + N_3(u,v,w) = f_3(x,y,t), \end{aligned} \tag{3.1}$$

with initial conditions

$$\begin{aligned} u(x,y,0) = g_1(x,y),\\ v(x,y,0) = g_2(x,y),\\ w(x,y,0) = g_3(x,y), \end{aligned} \tag{3.2}$$

where L_1, L_2 and L_3 are first order linear differential operators $L_1 = L_2 = L_3 = \frac{\partial}{\partial t}$ which are assumed to be easily invertible, R_1, R_2 and R_3 are the remaining linear operator, N_1, N_2 and N_3 represents the nonlinear terms and $f_1(x,y,t)$, $f_2(x,y,t)$ and $f_3(x,y,t)$ are known functions. The methodology consists of applying double Kamal-Shehu transform first on both sides of equations (3.1)

$$K_yS_t[L_1u(x,y,t)] + K_yS_t[R_1(u,v,w)] + K_yS_t[N_1(u,v,w)] = K_yS_t[f_1(x,y,t)],$$
$$K_yS_t[L_2u(x,y,t)] + K_yS_t[R_2(u,v,w)] + K_yS_t[N_2(u,v,w)] = K_yS_t[f_2(x,y,t)],$$
$$K_yS_t[L_3u(x,y,t)] + K_yS_t[R_3(u,v,w)] + K_yS_t[N_3(u,v,w)] = K_yS_t[f_3(x,y,t)].$$

Using the differentiation property of double Kamal-Shehu transform, we have

Application of inverse double Kamal-Shehu transform to equations (3.4) leads to

$$\begin{aligned}
u(x,y,t) &= K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K[g_1(x,y)] + \frac{\mu}{\delta}K_yS_t[f_1(x,y,t)]\right] \\
&- K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_yS_t\left[R_1(u,v,w) + N_1(u,v,w)\right]\right], \\
v(x,y,t) = &= K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K[g_2(x,y)] + \frac{\mu}{\delta}K_yS_t[f_2(x,y,t)]\right] \\
&- K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_yS_t\left[R_2(u,v,w) + N_2(u,v,w)\right]\right], \qquad (3.5) \\
w(x,y,t) = &= K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K[g_3(x,y)] + \frac{\mu}{\delta}K_yS_t[f_3(x,y,t)]\right] \\
&- K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_yS_t\left[R_3(u,v,w) + N_3(u,v,w)\right]\right].
\end{aligned}$$

The second step in double Kamal-Shehu decomposition method is that we represent solution as an infinite series:

$$\begin{aligned}
u(x,y,t) &= \sum_{i=0}^{\infty} u_i(x,y,t), \\
v(x,y,t) &= \sum_{i=0}^{\infty} v_i(x,y,t), \qquad (3.6) \\
w(x,y,t) &= \sum_{i=0}^{\infty} w_i(x,y,t),
\end{aligned}$$

and the nonlinear term can be decomposed as

$$N_1 = \sum_{i=0}^{\infty} A_i, \quad N_2 = \sum_{i=0}^{\infty} B_i, \quad N_3 = \sum_{i=0}^{\infty} C_i, \tag{3.7}$$

where A_i, B_i and C_i are Adomian polynomials denoted by formula

$$A_i = \frac{1}{i!}\frac{d^i}{d\lambda^i}\left[N_1 \sum_{i=0}^{\infty} \lambda^i u_i\right]_{\lambda=0}, \quad B_i = \frac{1}{i!}\frac{d^i}{d\lambda^i}\left[N_2 \sum_{i=0}^{\infty} \lambda^i v_i\right]_{\lambda=0}, \quad C_i = \frac{1}{i!}\frac{d^i}{d\lambda^i}\left[N_3 \sum_{i=0}^{\infty} \lambda^i w_i\right]_{\lambda=0}. \tag{3.8}$$

Substituting equations (3.6) and equations (3.7) in equations (3.5), we get

$$\begin{aligned}
\sum_{i=0}^{\infty} u(x,y,t) &= K_y^{-1}S_t^{-1}\Big[\frac{\mu}{\delta}K[g_1(x,y)] + \frac{\mu}{\delta}K_yS_t[f_1(x,y,t)]\Big] \\
&- K_y^{-1}S_t^{-1}\Big[\frac{\mu}{\delta}K_yS_t\Big[R_1\Big(\sum_{i=0}^{\infty} u_i, \sum_{i=0}^{\infty} v_i, \sum_{i=0}^{\infty} w_i\Big)\Big] + \frac{\mu}{\delta}K_yS_t[A_i]\Big], \\
\sum_{i=0}^{\infty} v(x,y,t) &= K_y^{-1}S_t^{-1}\Big[\frac{\mu}{\delta}K[g_2(x,y)] + \frac{\mu}{\delta}K_yS_t[f_2(x,y,t)]\Big] \\
&- K_y^{-1}S_t^{-1}\Big[\frac{\mu}{\delta}K_yS_t\Big[R_2\Big(\sum_{i=0}^{\infty} u_i, \sum_{i=0}^{\infty} v_i, \sum_{i=0}^{\infty} w_i\Big)\Big] + \frac{\mu}{\delta}K_yS_t[B_i]\Big], \\
\sum_{i=0}^{\infty} w(x,y,t) &= K_y^{-1}S_t^{-1}\Big[\frac{\mu}{\delta}K[g_3(x,y)] + \frac{\mu}{\delta}K_yS_t[f_3(x,y,t)]\Big] \\
&- K_y^{-1}S_t^{-1}\Big[\frac{\mu}{\delta}K_yS_t\Big[R_3\Big(\sum_{i=0}^{\infty} u_i, \sum_{i=0}^{\infty} v_i, \sum_{i=0}^{\infty} w_i\Big)\Big] + \frac{\mu}{\delta}K_yS_t[C_i]\Big].
\end{aligned} \tag{3.9}$$

On comparing both sides of the equations (3.9) and by using standard Adomian decomposition method (ADM), we then define the recurrence relations as

$$\begin{aligned}
u_0(x,t) &= K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K[g_1(x,y)] + \frac{\mu}{\delta}K_yS_t[f_1(x,y,t)]\right],\\
v_0(x,t) &= K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K[g_2(x,y)] + \frac{\mu}{\delta}K_yS_t[f_2(x,y,t)]\right],\\
w_0(x,t) &= K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K[g_3(x,y)] + \frac{\mu}{\delta}K_yS_t[f_3(x,y,t)]\right],\\
u_1(x,t) &= -K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_yS_t\big[R_1(u_0,v_0,w_0)\big] + \frac{\mu}{\delta}K_yS_t[A_0]\right],\\
v_1(x,t) &= -K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_yS_t\big[R_2(u_0,v_0,w_0)\big] + \frac{\mu}{\delta}K_yS_t[B_0]\right],\\
w_1(x,t) &= -K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_yS_t\big[R_3(u_0,v_0,w_0)\big] + \frac{\mu}{\delta}K_yS_t[C_0]\right],\\
u_2(x,t) &= -K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_yS_t\big[R_1(u_1,v_1,w_1)\big] + \frac{\mu}{\delta}K_yS_t[A_1]\right],\\
v_2(x,t) &= -K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_yS_t\big[R_2(u_1,v_1,w_1)\big] + \frac{\mu}{\delta}K_yS_t[B_1]\right],\\
w_2(x,t) &= -K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_yS_t\big[R_3(u_1,v_1,w_1)\big] + \frac{\mu}{\delta}K_yS_t[C_1]\right].
\end{aligned}$$

In more general, the recursive relation is given by

$$\begin{aligned}
u_{i+1}(x,y,t) &= -K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_yS_t\big[R_1(u_i,v_i,w_i)\big] + \frac{\mu}{\delta}K_yS_t[A_i]\right],\ i \geq 0,\\
v_{i+1}(x,y,t) &= -K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_yS_t\big[R_2(u_i,v_i,w_i)\big] + \frac{\mu}{\delta}K_yS_t[B_i]\right],\ i \geq 0,\\
w_{i+1}(x,y,t) &= -K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_yS_t\big[R_3(u_i,v_i,w_i)\big] + \frac{\mu}{\delta}K_yS_t[C_i]\right],\ i \geq 0.
\end{aligned}$$

The recurrence relation generates the solutions of (3.1) in series form given by

$$\begin{aligned}
u(x,y,t) &= u_0(x,y,t) + u_1(x,y,t) + u_2(x,y,t) + \cdots + u_i(x,y,t) + \cdots,\\
v(x,y,t) &= v_0(x,y,t) + v_1(x,y,t) + v_2(x,y,t) + \cdots + v_i(x,y,t) + \cdots,\\
w(x,y,t) &= w_0(x,y,t) + w_1(x,y,t) + w_2(x,y,t) + \cdots + w_i(x,y,t) + \cdots.
\end{aligned}$$

1. Applications

In order to illustrate the applicability and efficiency of the double Kamal-Shehu decomposition method, we apply this method for solving some kinds of nonlinear system partial differential equations.

Example 4.1. *Consider the following nonlinear partial differential equation*

$$u_{tt}(x,t) - u_{xx}(x,t) + u^2(x,t) = x^2t^2, \qquad t>0, \tag{4.1}$$

subject to the conditions

$$u(x,0)=0, \qquad u_t(x,0)=x, \tag{4.2}$$

$$u(0,t)=0, \qquad u_x(0,t)=t. \tag{4.3}$$

Taking the double Kamal-Shehu transform first on both sides of (4.1), then by using the differentiation property of double Kamal-Shehu transform, we have

$$\begin{aligned}\frac{\delta^2}{\mu^2}F(\gamma,\delta,\mu) \quad &- \quad \frac{\delta}{\mu}K[u(x,0)] - K[u_t(x,0)] - \frac{1}{\gamma^2}F(\gamma,\delta,\mu) + \frac{1}{\gamma}S[u(0,t)] + S[u_x(0,t)] \\ &= \quad K_xS_t[x^2t^2] - K_xS_t[u^2]. \end{aligned} \tag{4.4}$$

Application of single Kamal transform to (4.2), single Shehu transform to (4.3) and substitute in (4.4), we have

$$\Big(\frac{\gamma^2\delta^2-\mu^2}{\gamma^2\mu^2}\Big)F(\gamma,\delta,\mu) = \gamma^2 - \frac{\mu^2}{\delta^2} + \frac{4\gamma^3\mu^3}{\delta^3} - K_xS_t[u^2], \tag{4.5}$$

by simple computation, we get

$$F(\gamma,\delta,\mu) = \frac{\gamma^2\mu^2}{\delta^2} + \frac{\gamma^2\mu^2}{(\gamma^2\delta^2-\mu^2)}\frac{4\gamma^3\mu^3}{\delta^3} - \frac{\gamma^2\mu^2}{(\gamma^2\delta^2-\mu^2)}K_xS_t[u^2]. \tag{4.6}$$

Taking the inverse double Kamal-Shehu transform in (4.6), our required recursive relation is given by

$$u(x,t) = xt + K_x^{-1}S_t^{-1}\Big[\frac{\gamma^2\mu^2}{(\gamma^2\delta^2-\mu^2)}\frac{4\gamma^3\mu^3}{\delta^3} - \frac{\gamma^2\mu^2}{(\gamma^2\delta^2-\mu^2)}K_xS_t[u^2]\Big]. \tag{4.7}$$

The double Kamal-Shehu decomposition method assumes a series solution of the function $u(x,t)$ is given by

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t). \tag{4.8}$$

Using Eq. (4.8) into Eq. (4.7), we get

$$\sum_{i=0}^{\infty} u_i(x,t) = xt + K_x^{-1}S_t^{-1}\left[\frac{\gamma^2\mu^2}{(\gamma^2\delta^2-\mu^2)}\frac{4\gamma^3\mu^3}{\delta^3} - \frac{\gamma^2\mu^2}{(\gamma^2\delta^2-\mu^2)}K_xS_t\Big[\sum_{i=0}^{\infty}A_i(u)\Big]\right], \tag{4.9}$$

where A_i are Adomian polynomials that represent nonlinear terms.
So Adomian polynomials are given as follows:

$$\sum_{i=0}^{\infty} A_i(u) = u^2. \tag{4.10}$$

The few components of the Adomian polynomials are given as follow:

$$A_0(u) = u_0^2, \;\; A_1(u) = 2u_0u_1, \;\; ..., \;\; A_i(u) = \sum_{r=0}^{i} u_r u_{(i-r)}.$$

From Eqs. (4.9) and (4.10), we obtain

$$\begin{aligned} u_0 &= xt, \\ u_{i+1}(x,t) &= K_x^{-1}S_t^{-1}\left[\frac{\gamma^2\mu^2}{(\gamma^2\delta^2-\mu^2)}\frac{4\gamma^3\mu^3}{\delta^3} - \frac{\gamma^2\mu^2}{(\gamma^2\delta^2-\mu^2)}K_xS_t\left[\sum_{i=0}^{\infty}A_i(u)\right]\right], \; i \geq 0. \end{aligned}$$

Then the first few components of $u_i(x,t)$ follow immediately upon setting

$$\begin{aligned} u_1(x,t) &= K_x^{-1}S_t^{-1}\Big[\frac{\gamma^2\mu^2}{(\gamma^2\delta^2-\mu^2)}\frac{4\gamma^3\mu^3}{\delta^3} - \frac{\gamma^2\mu^2}{(\gamma^2\delta^2-\mu^2)}K_xS_t\big[A_0(u)\big]\Big] \\ &= K_x^{-1}S_t^{-1}\Big[\frac{\gamma^2\mu^2}{(\gamma^2\delta^2-\mu^2)}\frac{4\gamma^3\mu^3}{\delta^3} - \frac{\gamma^2\mu^2}{(\gamma^2\delta^2-\mu^2)}K_xS_t\big[x^2t^2\big]\Big] \\ &= K_x^{-1}S_t^{-1}\Big[\frac{\gamma^2\mu^2}{(\gamma^2\delta^2-\mu^2)}\frac{4\gamma^3\mu^3}{\delta^3} - \frac{\gamma^2\mu^2}{(\gamma^2\delta^2-\mu^2)}\frac{4\gamma^3\mu^3}{\delta^3}\Big] \\ &= K_x^{-1}S_t^{-1}\big[0\big] \\ &= 0. \end{aligned}$$

Similarly, $u_2(x,t) = 0$ and so on for rest terms.
Therefore, the exact solution obtained by double Kamal-Shehu decomposition method is given as follows:

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t) = xt. \tag{4.11}$$

Which is the required solution.

Example 4.2. *Consider the system of nonlinear partial differential equations*

$$\begin{aligned} u_t + u_x v_x &= 2, \\ v_t + u_x v_x &= 0, \end{aligned} \tag{4.12}$$

with initial conditions

$$\begin{aligned} u(x,0) &= x, \\ v(x,0) &= x. \end{aligned} \tag{4.13}$$

Applying the double Kamal-Shehu transform to both sides of equations (4.12), we have

$$\begin{aligned} &\tfrac{\delta}{\mu}U(\gamma,\delta,\mu) - K[u(x,0)] = K_x S_t[2] - K_x S_t[u_x v_x], \\ &\tfrac{\delta}{\mu}V(\gamma,\delta,\mu) - K[v(x,0)] = -K_x S_t[u_x v_x]. \end{aligned} \tag{4.14}$$

Application of single Kamal transform to (4.13) and substitute in (4.14), we have

$$\begin{aligned} U(\gamma,\delta,\mu) &= \tfrac{\gamma^2\mu}{\delta} + \tfrac{2\gamma\mu^2}{\delta^2} - \tfrac{\mu}{\delta}K_x S_t[u_x v_x], \\ V(\gamma,\delta,\mu) &= \tfrac{\gamma^2\mu}{\delta} - \tfrac{\mu}{\delta}K_x S_t[u_x v_x]. \end{aligned} \tag{4.15}$$

By taking the inverse double Kamal-Shehu transform in (4.15), our required recursive relation is given by

$$\begin{aligned} u(x,t) &= x + 2t - K_x^{-1}S_t^{-1}\left[\tfrac{\mu}{\delta}K_x S_t[u_x v_x]\right], \\ v(x,t) &= x - K_x^{-1}S_t^{-1}\left[\tfrac{\mu}{\delta}K_x S_t[u_x v_x]\right]. \end{aligned} \tag{4.16}$$

The recursive relations are

$$\begin{aligned} u_0(x,t) &= x + 2t, \\ u_{i+1}(x,t) &= -K_x^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_x S_t\left[\sum_{i=0}^{\infty} C_i(u,v)\right]\right], \ i \geq 0, \\ v_0(x,t) &= x, \\ v_{i+1}(x,t) &= -K_x^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_x S_t\left[\sum_{i=0}^{\infty} D_i(u,v)\right]\right], \ i \geq 0, \end{aligned} \tag{4.17}$$

where $C_i(u, v)$ and $D_i(u, v)$ are Adomian polynomials representing the nonlinear terms [12]. The few components of Adomian polynomials are given as follow

$$
\begin{aligned}
C_0(u,v) &= u_{0x}v_{0x}, \\
C_1(u,v) &= u_{0x}v_{1x} + u_{1x}v_{0x}, \\
C_2(u,v) &= u_{0x}v_{2x} + u_{1x}v_{1x} + u_{2x}v_{0x}, \\
C_3(u,v) &= u_{0x}v_{3x} + u_{1x}v_{2x} + u_{2x}v_{1x} + u_{3x}v_{0x}, \\
&\vdots \\
C_n(u,v) &= \sum_{i=0}^{n} u_{ix}v_{(n-i)x}, \\
D_0(u,v) &= u_{0x}v_{0x}, \\
D_1(u,v) &= u_{0x}v_{1x} + u_{1x}v_{0x}, \\
D_2(u,v) &= u_{0x}v_{2x} + u_{1x}v_{1x} + u_{2x}v_{0x}, \\
D_3(u,v) &= u_{0x}v_{3x} + u_{1x}v_{2x} + u_{2x}v_{1x} + u_{3x}v_{0x}, \\
&\vdots \\
D_n(u,v) &= \sum_{i=0}^{n} u_{ix}v_{(n-i)x}.
\end{aligned}
$$

Using the derived Adomian polynomials into (4.17), we obtain

$$
\begin{aligned}
u_0(x,t) &= x + 2t, \\
v_0(x,t) &= x, \\
u_1(x,t) &= -K_x^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_xS_t\left[C_0(u,v)\right]\right] \\
&= -K_x^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_xS_t\left[u_{0x}v_{0x}\right]\right] \\
&= -K_x^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_xS_t\left[1\right]\right] \\
&= -K_x^{-1}S_t^{-1}\left[\frac{\gamma\mu^2}{\delta^2}\right] \\
&= -t,
\end{aligned}
$$

$$
\begin{aligned}
v_1(x,t) &= -K_x^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_xS_t\big[D_0(u,v)\big]\right] \\
&= -K_x^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_xS_t\big[u_{0x}v_{0x}\big]\right] \\
&= -K_x^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_xS_t\big[1\big]\right] \\
&= -K_x^{-1}S_t^{-1}\left[\frac{\gamma\mu^2}{\delta^2}\right] \\
&= -t, \\
u_2(x,t) &= -K_x^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_xS_t\big[C_1(u,v)\big]\right] \\
&= -K_x^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_xS_t\big[u_{0x}v_{1x}+u_{1x}v_{0x}\big]\right] \\
&= -K_x^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_xS_t\big[0\big]\right] \\
&= 0, \\
v_2(x,t) &= -K_x^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_xS_t\big[D_1(u,v)\big]\right] \\
&= -K_x^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_xS_t\big[u_{0x}v_{1x}+u_{1x}v_{0x}\big]\right] \\
&= -K_x^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_xS_t\big[0\big]\right], \\
&= 0.
\end{aligned}
$$

Similarly, $u_3(x,t) = v_3(x,t) = 0$ and so on for rest terms.
Therefore, the solution of system (4.12) obtained by double Kamal-Shehu decomposition method are given by

$$
\begin{aligned}
u(x,t) &= \sum_{i=0}^{\infty} u_i(y,t) = x+t, \\
v(x,t) &= \sum_{i=0}^{\infty} v_i(y,t) = x-t.
\end{aligned}
$$

Example 4.3. *Consider the system of nonlinear partial differential equations*

$$\begin{cases} u_t + u_x v_x - w_y = 1, \\ v_t - v_x w_x + u_y = 1, \\ w_t - u_x w_x + v_y = 1, \end{cases} \tag{4.18}$$

with initial conditions

$$\begin{cases} u(x,y,0) = x + y, \\ v(x,y,0) = x - y, \\ w(x,y,0) = -x + y. \end{cases} \tag{4.19}$$

Taking the double Kamal-Shehu transform to both sides of equations (4.18), we have

$$\begin{aligned} \frac{\delta}{\mu}u(x,\gamma,\delta,\mu) - K[u(x,y,0)] &= \frac{\gamma\mu}{\delta} + K_y S_t[w_y - u_x v_x], \\ \frac{\delta}{\mu}v(x,\gamma,\delta,\mu) - K[v(x,y,0)] &= \frac{\gamma\mu}{\delta} - K_y S_t[u_y + v_x w_x], \\ \frac{\delta}{\mu}w(x,\gamma,\delta,\mu) - K[w(x,y,0)] &= \frac{\gamma\mu}{\delta} - K_y S_t[v_y - u_x w_x]. \end{aligned} \tag{4.20}$$

Application of single Kamal transform to (4.19) then substitute in (4.20) and rearranging the terms, we have

$$\begin{aligned} u(x,\gamma,\delta,\mu) &= \frac{\gamma\mu^2}{\delta^2} + \frac{\mu}{\delta}(x\gamma + \gamma^2) + \frac{\mu}{\delta}K_y S_t[w_y - u_x v_x], \\ v(x,\gamma,\delta,\mu) &= \frac{\gamma\mu^2}{\delta^2} + \frac{\mu}{\delta}(x\gamma - \gamma^2) - \frac{\mu}{\delta}K_y S_t[u_y + v_x w_x], \\ w(x,\gamma,\delta,\mu) &= \frac{\gamma\mu^2}{\delta^2} + \frac{\mu}{\delta}(-x\gamma + \gamma^2) - \frac{\mu}{\delta}K_y S_t[v_y - u_x w_x]. \end{aligned} \tag{4.21}$$

By taking the inverse double Kamal-Shehu transform in (4.27), we get

$$\begin{aligned} u(x,y,t) &= t + x + y + K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_y S_t[w_y - u_x v_x]\right], \\ v(x,y,t) &= t + x - y - K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_y S_t[u_y + v_x w_x]\right], \\ w(x,y,t) &= t - x + y - K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_y S_t[v_y - u_x w_x]\right]. \end{aligned} \tag{4.22}$$

The recursive relations are

$$\begin{aligned}
u_0(x,y,t) &= t+x+y,\\
u_{i+1}(x,y,t) &= K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_yS_t\left[\sum_{i=0}^{\infty}w_{iy}-\sum_{i=0}^{\infty}E_i(u,v)\right]\right],\ i\geq 0,\\
v_0(x,y,t) &= t+x-y,\\
v_{i+1}(x,y,t) &= -K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_yS_t\left[\sum_{i=0}^{\infty}u_{iy}+\sum_{i=0}^{\infty}F_i(v,w)\right]\right],\ i\geq 0,\\
w_0(x,y,t) &= t-x+y,\\
w_{i+1}(x,y,t) &= -K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_yS_t\left[\sum_{i=0}^{\infty}v_{iy}-\sum_{i=0}^{\infty}G_i(u,w)\right]\right],\ i\geq 0,
\end{aligned}\tag{4.23}$$

where $E_i(u,v)$, $F_i(v,w)$ and $G_i(u,w)$ are Adomian polynomials representing the nonlinear terms [12] in above equations. The few components of Adomian polynomials are given as follow

$$\begin{aligned}
E_0(u,v) &= u_{0x}v_{0x},\\
E_1(u,v) &= u_{1x}v_{0x}+u_{0x}v_{1x},\\
&\vdots\\
E_i(u,v) &= \sum_{n=0}^{i}u_{nx}v_{(i-n)x},\\
F_0(v,w) &= v_{0x}w_{0x},\\
F_1(v,w) &= v_{1x}w_{0x}+v_{0x}w_{1x},\\
&\vdots\\
F_i(v,w) &= \sum_{n=0}^{i}v_{nx}w_{(i-n)x},\\
G_0(u,w) &= u_{0x}w_{0x},\\
G_1(u,w) &= u_{1x}w_{0x}+u_{0x}w_{1x},\\
&\vdots\\
G_i(u,w) &= \sum_{n=0}^{i}u_{nx}w_{(i-n)x}.
\end{aligned}$$

In view of this recursive relations we obtained other components of the solution as follows:

$$\begin{aligned} u_1(x,y,t) &= K_y^{-1}S_t^{-1}\Big[\frac{\mu}{\delta}K_yS_t[w_{0y}-E_0(u,v)]\Big] = K_y^{-1}S_t^{-1}\Big[\frac{\mu}{\delta}K_yS_t[w_{0y}-u_{0x}v_{0x}]\Big] \\ &= K_y^{-1}S_t^{-1}\Big[0\Big] = 0, \\ v_1(x,y,t) &= -K_y^{-1}S_t^{-1}\Big[\frac{\mu}{\delta}K_yS_t[u_{0y}+F_0(w,u)]\Big] = -K_y^{-1}S_t^{-1}\Big[\frac{\mu}{\delta}K_yS_t[u_{0y}+v_{0x}w_{0x}]\Big] \\ &= -K_y^{-1}S_t^{-1}\Big[0\Big] = 0, \\ w_1(x,y,t) &= -K_y^{-1}S_t^{-1}\Big[\frac{\mu}{\delta}K_yS_t[v_{0y}-G_0(u,v)]\Big] = -K_y^{-1}S_t^{-1}\Big[\frac{\mu}{\delta}K_yS_t[v_{0y}-u_{0x}w_{0x}]\Big] \\ &= -K_y^{-1}S_t^{-1}\Big[0\Big] = 0. \end{aligned}$$

So, $u_2(x,y,t) = v_2(x,y,t) = w_2(x,y,t) = 0$ and so on for rest terms.
Therefore, the solution of system (4.18) are given below

$$\begin{aligned} u(x,y,t) &= \sum_{i=0}^{\infty} u_i(x,y,t) = t+x+y, \\ v(x,y,t) &= \sum_{i=0}^{\infty} v_i(x,y,t) = t+x-y, \\ w(x,y,t) &= \sum_{i=0}^{\infty} w_i(x,y,t) = t-x+y. \end{aligned}$$

Example 4.4. *Consider the system of nonlinear partial differential equations*

$$\begin{cases} u_t + u_yv_x = 1+e^t, \\ v_t + v_yw_x = 1-e^{-t}, \\ w_t + u_yw_y = 1-e^{-t}, \end{cases} \tag{4.24}$$

with initial conditions

$$\begin{cases} u(x,y,0) = 1+x+y, \\ v(x,y,0) = 1+x-y, \\ w(x,y,0) = 1-x+y. \end{cases} \tag{4.25}$$

Taking the double Kamal-Shehu transform to both sides of equations (4.24), we have

$$\begin{aligned} \frac{\delta}{\mu}u(x,\gamma,\delta,\mu) - K[u(x,y,0)] &= \frac{\gamma\mu}{\delta} + \frac{\gamma\mu}{\delta-\mu} - K_yS_t[u_yv_x], \\ \frac{\delta}{\mu}v(x,\gamma,\delta,\mu) - K[v(x,y,0)] &= \frac{\gamma\mu}{\delta} - \frac{\gamma\mu}{\delta+\mu} - K_yS_t[v_yw_x], \\ \frac{\delta}{\mu}w(x,\gamma,\delta,\mu) - K[w(x,y,0)] &= \frac{\gamma\mu}{\delta} - \frac{\gamma\mu}{\delta+\mu} - K_yS_t[u_yw_y]. \end{aligned} \tag{4.26}$$

Application of single Kamal transform to (4.25) then substitute in (4.26) and rearranging the terms, we have

$$\begin{aligned} u(x,\gamma,\delta,\mu) &= \frac{\gamma\mu}{\delta} + \frac{\gamma\mu x}{\delta} + \frac{\gamma^2\mu}{\delta} + \frac{\gamma\mu^2}{\delta^2} + \frac{\gamma\mu^2}{\delta(\delta-\mu)} - \frac{\mu}{\delta}K_yS_t[u_yv_x], \\ v(x,\gamma,\delta,\mu) &= \frac{\gamma\mu}{\delta} + \frac{\gamma\mu x}{\delta} - \frac{\gamma^2\mu}{\delta} + \frac{\gamma\mu^2}{\delta^2} - \frac{\gamma\mu^2}{\delta(\delta+\mu)} - \frac{\mu}{\delta}K_yS_t[v_yw_x], \\ w(x,\gamma,\delta,\mu) &= \frac{\gamma\mu}{\delta} - \frac{\gamma\mu x}{\delta} + \frac{\gamma^2\mu}{\delta} + \frac{\gamma\mu^2}{\delta^2} - \frac{\gamma\mu^2}{\delta(\delta+\mu)} - \frac{\mu}{\delta}K_yS_t[u_yw_y], \end{aligned}$$

by simple computation, we get

$$\begin{aligned} u(x,\gamma,\delta,\mu) &= \frac{\gamma\mu}{\delta-\mu}+\frac{\gamma^2\mu}{\delta}+\frac{\gamma\mu^2}{\delta^2}+\frac{\gamma\mu x}{\delta}-\frac{\mu}{\delta}K_yS_t[u_yv_x], \\ v(x,\gamma,\delta,\mu) &= \frac{\gamma\mu}{\delta+\mu}-\frac{\gamma^2\mu}{\delta}+\frac{\gamma\mu^2}{\delta^2}+\frac{\gamma\mu x}{\delta}-\frac{\mu}{\delta}K_yS_t[v_yw_x], \\ w(x,\gamma,\delta,\mu) &= \frac{\gamma\mu}{\delta+\mu}+\frac{\gamma^2\mu}{\delta}+\frac{\gamma\mu^2}{\delta^2}-\frac{\gamma\mu x}{\delta}-\frac{\mu}{\delta}K_yS_t[u_yw_y]. \end{aligned} \tag{4.27}$$

By taking the inverse double Kamal-Shehu transform in (4.27), we get

$$\begin{aligned} u(x,y,t) &= e^t+y+t+x-K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_yS_t[u_yv_x]\right], \\ v(x,y,t) &= e^{-t}-y+t+x-K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_yS_t[v_yw_x]\right], \\ w(x,y,t) &= e^{-t}+y+t-x-K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_yS_t[u_yw_y]\right]. \end{aligned}$$

$$\begin{aligned} u_0(x,y,t) &= e^t+y+t+x, \\ u_{i+1}(x,y,t) &= -K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_yS_t\left[\sum_{i=0}^{\infty}H_i(u,v)\right]\right],\ i\geq 0, \\ v_0(x,y,t) &= e^{-t}-y+t+x, \\ v_{i+1}(x,y,t) &= -K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_yS_t\left[\sum_{i=0}^{\infty}I_i(v,w)\right]\right],\ i\geq 0, \\ w_0(x,y,t) &= e^{-t}+y+t-x, \\ w_{i+1}(x,y,t) &= -K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_yS_t\left[\sum_{i=0}^{\infty}J_i(u,w)\right]\right],\ i\geq 0, \end{aligned}$$

The recursive relations are

where $H_i(u,v)$, $I_i(v,w)$ and $J_i(u,w)$ are Adomian polynomials representing the nonlinear terms [12] in above equations. The few components of Adomian polynomials are given as follow

$$\begin{aligned} H_0(u,v) &= u_{0y}v_{0x}, \\ H_1(u,v) &= u_{1y}v_{0x}+u_{0y}v_{1x}, \\ &\vdots \\ H_i(u,v) &= \sum_{n=0}^{i}u_{ny}v_{(i-n)x}, \\ I_0(v,w) &= v_{0y}w_{0x}, \\ I_1(v,w) &= v_{1y}w_{0x}+v_{0y}w_{1x}, \\ &\vdots \\ I_i(v,w) &= \sum_{n=0}^{i}v_{ny}w_{(i-n)x}, \\ J_0(u,w) &= u_{0y}w_{0y}, \\ J_1(u,w) &= u_{1y}w_{0y}+u_{0y}w_{1y}, \\ &\vdots \\ J_i(u,w) &= \sum_{n=0}^{i}u_{ny}w_{(i-n)y}. \end{aligned}$$

In view of this recursive relations we obtained other components of the solution as follows

$$\begin{aligned}
u_1(x,y,t) &= -K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_yS_t[H_0(u,v)]\right] = -K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_yS_t[u_{0y}v_{0x}]\right] \\
&= -K_y^{-1}S_t^{-1}\left[\frac{\gamma\mu^2}{\delta^2}\right] = -t, \\
v_1(x,y,t) &= -K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_yS_t[I_0(v,w)]\right] = -K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_yS_t[v_{0y}w_{0x}]\right] \\
&= -K_y^{-1}S_t^{-1}\left[\frac{\gamma\mu^2}{\delta^2}\right] = -t, \\
w_1(x,y,t) &= -K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_yS_t[J_0(u,w)]\right] = -K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_yS_t[u_{0y}w_{0y}]\right] \\
&= -K_y^{-1}S_t^{-1}\left[\frac{\gamma\mu^2}{\delta^2}\right] = -t, \\
u_2(x,y,t) &= -K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_yS_t[H_1(u,v)]\right] = -K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_yS_t[u_{1y}v_{0x}+u_{0y}v_{1x}]\right] = 0, \\
v_2(x,y,t) &= -K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_yS_t[I_1(v,w)]\right] = -K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_yS_t[v_{1y}w_{0x}+v_{0y}w_{1x}]\right] = 0, \\
w_2(x,y,t) &= -K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_yS_t[j_1(u,w)]\right] = -K_y^{-1}S_t^{-1}\left[\frac{\mu}{\delta}K_yS_t[u_{1y}w_{0y}+u_{0y}w_{1y}]\right] = 0.
\end{aligned}$$

So, $u_3(x, y, t) = v_3(x, y, t) = w_3(x, y, t) = 0$ and so on for rest terms.

Therefore, the solution of system (4.24) are given below

$$\begin{aligned}
u(x,y,t) &= \sum_{i=0}^{\infty} u_i(x,y,t) = e^t + y + t + x - t \\
&= e^t + y + x, \\
v(x,y,t) &= \sum_{i=0}^{\infty} v_i(x,y,t) = e^{-t} - y + t + x - t \\
&= e^{-t} - y + x, \\
w(x,y,t) &= \sum_{i=0}^{\infty} w_i(x,y,t) = e^{-t} + y + t - x - t \\
&= e^{-t} + y - x.
\end{aligned}$$

Which is same as solution obtained by variational iteration method (VIM) [11].

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